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Travelling Wave Solutions and Conservation Laws of the Korteweg-de Vries-Burgers Equation with Power Law Nonlinearity

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ABSTRACT

In this work we present travelling wave solutions and conservation laws of the Korteweg-de Vries-Burgers equation with power law nonlinearity. This is a modification of the Korteweg-de Vries-Burgers equation which was derived for a wide class of nonlinear system in weak nonlinearity and long wavelength approximation. Lie symmetry method along with Kudryashov's approach are used to obtain exact solutions while the new theorem due to Ibragimov is used to construct conservation laws.

Keywords: Travelling wave solutions, conservation laws and Korteweg-de Vries-Burgers equation with power law nonlinearity.

1. Introduction

In this paper we study the Korteweg-de Vries-Burgers equation with power law nonlinearity given by

$$u_t + \alpha u^n u_x - \beta u_{xx} + \gamma u_{xxx} = 0. \tag{1}$$

When $n = 1$ this equation reduces to the Korteweg-de Vries-Burgers equation which was derived for a wide class of nonlinear system in weak nonlinearity and long wavelength approximation Sayed and Danaf (2002).

Nonlinear partial differential equations are widely used as models to describe physical phenomena in different fields of applied sciences, such as fluid mechanics, solid state physics, plasma physics and plasma waves. A basic mathematical problem for such models is to obtain closed form solutions. Different methods (see Mhlanga and Khalique (2013) and references therein) have been developed and used to find different kinds of solutions of such physical models. Some of the methods used include the variational iteration method, the exp-function method, the inverse scattering transform method, the sine-cosine method, the Lie group method and the $(G'/G)-$expansion method.

In this work firstly we use the method of Lie symmetry analysis along with Kudryashov's approach to find exact solutions of (1). Lie's continuous symmetry groups have applications in many fields such as invariant theory, control theory, classical mechanics, relativity, etc. For the theory and application of the Lie symmetry analysis methods, see for example, Olver (1993) and Ibragimov (1994–1996).

On the other hand, conservation laws play a vital role in the study of differential equations and in many physical phenomena. In mathematics, it provides one of the basic principles in formulating and investigating models Zhijie and Yiping (2014). The high number of conservation laws for a partial differential equation guarantees that the partial differential equation is strongly integrable and can be linearized or explicitly solved Bluman and Kumei (1989). There are different methods for the construction of conservation laws Naz et al. (2008) and Hereman and Nuseir (1997) and Wolf (2002). These include amongst others the direct method, Noether's approach, characteristic method, variational approach, variational approach on the space of solutions of the differential equation, symmetry and conservation law relation, partial Noether approach and Noether approach for a system and its adjoint. In the present work, the new theorem due to Ibragimov Ibragimov (2007) is used to construct conservation laws of (1).

2. Similarity reduction and exact solutions

2.1 Similarity reduction of (1)

First we find the Lie point symmetries of (1) using the Lie algorithm Olver (1993). These Lie point symmetries are then used to transform (1) into an ordinary differential equation. Kudryashov's approach Kudryashov (2012) is then applied to the ordinary differential equation and as a result we obtain the exact solutions of (1).

The symmetries of (1) will be generated by the vector field of the form

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2}$$

Applying the third prolongation, $\mathrm{pr}^{(3)}X$, Olver (1993) to (1) leads to the overdetermined system of linear partial differential equations

$$\begin{aligned}
&\xi^1_x = 0,\\
&\xi^1_u = 0,\\
&\xi^2_{uuu} = 0,\\
&\beta\xi^2_{uu} - 3\gamma\xi^2_{xuu} + \gamma\eta_{uuu} = 0,\\
&2u^n\alpha\gamma\xi^2_{uu} - \beta^2\xi^2_{uu} + 3\beta\gamma\xi^2_{xuu} - \beta\gamma\eta_{uuu} = 0,\\
&4u^n\alpha\gamma\xi^2_{uu} - \beta^2\xi^2_{uu} + 3\beta\gamma\xi^2_{xuu} - \beta\gamma\eta_{uuu} = 0,\\
&6u^n\alpha\gamma\xi^2_{uu} - \beta^2\xi^2_{uu} + 3\beta\gamma\xi^2_{xuu} - \beta\gamma\eta_{uuu} = 0,\\
&2\beta\xi^2_{xu} - \beta\eta_{uu} - 3\gamma\xi^2_{xxu} + 3\gamma\eta_{xuu} = 0,\\
&\beta\eta_{xx} - \alpha u^n\eta_x - \gamma\eta_{xxx} - \eta_t = 0,\\
&6\gamma\eta_{xuu} - 6\gamma\xi^2_{xxu} + 4\beta\xi^2_{xu} - 2\beta\eta_{uu} + 3\alpha u^n\xi^2_u = 0,\\
&3\alpha u^n\xi^2_u + 2\beta\xi^2_{xu} - 3\gamma\xi^2_{xxu} + 3\gamma\eta_{xuu} - \beta\eta_{uu} = 0,\\
&5\alpha\beta u^n\xi^2_u - 9\alpha\gamma u^n\xi^2_{xu} + 3\alpha\gamma u^n\eta_{uu} + 4\beta^2\xi^2_{xu} - 2\beta^2\eta_{uu} - 6\beta\gamma\xi^2_{xxu}\\
&+6\beta\gamma\eta_{xuu} = 0,\\
&u\xi^2_t - 2\alpha uu^n\xi^2_x - \alpha nu^n\eta - 3\gamma u\eta_{xxu} - \beta u\xi^2_{xx} + 2\beta u\eta_{xu} + \gamma u\xi^2_{xxx} = 0,\\
&2\alpha\beta u^n\xi^2_u - 9\alpha\gamma u^n\xi^2_{xu} + 4\beta^2\xi^2_{xu} + 3\alpha\gamma u^n\eta_{uu} - 2\beta^2\eta_{uu} - 6\beta\gamma\xi^2_{xxu}\\
&+6\beta\gamma\eta_{xuu} = 0,\\
&3\alpha^2\gamma u^{2n}\xi^2_u - 2\alpha\beta^2u^n\xi^2_u + 9\alpha\beta\gamma u^n\xi^2_{xu} - 3\alpha\beta\gamma u^n\eta_{uu} - 2\beta^3\xi^2_{xu} + \beta^3\eta_{uu}\\
&+3\beta^2\gamma\xi^2_{xxu} - 3\beta^2\gamma\eta_{xuu} = 0,\\
&3\gamma u\eta_{xxu} - u\xi^2_x - \gamma u\xi^2_{xxx} + \alpha nu^n\eta + \alpha nu^n\xi^1_t + \beta u\xi^2_{xx} - 2\beta u\eta_{xu}\\
&-\alpha uu^n\xi^2_x = 0,
\end{aligned}$$

$$\beta u\xi_t^2 - \alpha\beta n u^n \eta - \alpha\beta u u^n \xi_x^2 + 3\alpha\gamma u u^n \xi_{xx}^2 - \beta^2 u \xi_{xx}^2 - 3\alpha\gamma u u^n \eta_{xu}$$
$$+2\beta^2 u \eta_{xu} + \beta\gamma u \xi_{xxx}^2 - 3\beta\gamma \eta_{xxu} = 0.$$

Solving this system we obtain two translation symmetries

$$X_1 = \frac{\partial}{\partial t} \quad \text{and} \quad X_2 = \frac{\partial}{\partial x}. \tag{3}$$

Considering the linear combination symmetry $X = kX_1 - \omega X_2$, one can easily solve the corresponding Lagrange system to obtain an invariant $z = kx + \omega t$, where k and ω are constants, and so the group-invariant solution of (1) is of the form

$$u = F(z). \tag{4}$$

Substituting the value of u from (4) into (1) we obtain the third-order nonlinear ordinary differential equation

$$\gamma k^3 F^{(3)}(z) - \beta k^2 F''(z) + \alpha k F(z)^n F'(z) + \omega F'(z) = 0. \tag{5}$$

Integrating (5) once with respect to the variable z and taking the constant of integration to be zero reduces it to the second-order nonlinear ordinary differential equation

$$\gamma k^3 F''(z) - \beta k^2 F'(z) + \frac{\alpha k F(z)^{n+1}}{n+1} + \omega F(z) = 0. \tag{6}$$

We now use the transformation

$$F(z) = G(z)^{\frac{1}{n}} \tag{7}$$

to transform (6) into the nonlinear ordinary differential equation for G, viz.,

$$n^2 \omega G(z)^2 + n^3 \omega G(z)^2 + \alpha\, k n^2 G(z)^3 - \gamma\, k^3 (n^2 - 1) G'(z)^2$$
$$+k^2 n(n+1) G(z)\{-\beta G'(z) + k\gamma G''(z)\} = 0. \tag{8}$$

2.2 Exact solutions of (1)

In this section we employ Kudryashov's method Kudryashov (2012) to find exact solutions of (1). Let us consider the solutions of (8) in the form

$$G(z) = \sum_{i=0}^{N} a_i Q(z)^i, \tag{9}$$

where N is a positive integer that can be determined by the balancing procedure as in Kudryashov (2012) and a_i's are constants to be determined. The function $Q(z)$ satisfies the first-order equation

$$Q'(z) = Q(z)^2 - Q(z) \tag{10}$$

whose solution is given by

$$Q(z) = \frac{1}{1+e^z}. \tag{11}$$

In our case the balancing procedure yields $N = 2$, and so the solution of (8) is of the form

$$G(z) = a_0 + a_1 Q(z) + a_2 Q(z)^2. \tag{12}$$

Substituting (12) into (8) and making use of the equation (10) and then equating the coefficients of the powers of Q to zero, we obtain an algebraic system of equations in terms of a_i $(i = 0, 1, 2)$. Solving this system of algebraic equations with the aid of Maple, one possible set of values of the constants is

$$\begin{aligned}
a_0 &= -\frac{2\,\gamma\,k^2(n+1)(n+2)}{\alpha n^2},\\
a_1 &= \frac{4\,\gamma\,k^2(n+1)(n+2)}{\alpha n^2},\\
a_2 &= -\frac{2\,\gamma\,k^2(n+1)(n+2)}{\alpha n^2},\\
\beta &= \frac{\gamma\,k(n+4)}{n},\\
\omega &= -\frac{2\,\gamma\,k^2}{n^2}\left(4n - 5kn - 18k + 16\right).
\end{aligned}$$

As a result, a solution of the Korteweg-de Vries-Burgers equation with power law nonlinearity (1) is of the form

$$F(z) = \left\{a_0 + a_1\left(\frac{1}{1+e^z}\right) + a_2\left(\frac{1}{1+e^z}\right)^2\right\}^{1/n}, \tag{13}$$

where $z = kx + \omega t$ and the values of a_0, a_1 and a_2 are as given above.

3. Construction of conservation laws for (1)

In this section we construct conservation laws for the KdVB equation (1). The new conservation theorem due to Ibragimov Ibragimov (2007) will be used.

First we compute the adjoint of equation (1) which is given by

$$\frac{\delta}{\delta u}[v(u_t + \alpha u^n u_x - \beta u_{xx} + \gamma u_{xxx})] = 0, \tag{14}$$

where $\delta/\delta u$ is the Euler-Lagrange operator defined as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}} \tag{15}$$

and the total derivative operators D_t and D_x are given by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots , \end{aligned} \tag{16}$$

respectively. Expanding Eq.(14) using (15) and (16) we obtain the adjoint equation

$$v_t + \alpha u^n v_x + \beta v_{xx} + \gamma v_{xxx} = 0. \tag{17}$$

In this case the system formed by the KdVB equation (1) together with its adjoint equation is given by

$$u_t + \alpha u^n u_x - \beta u_{xx} + \gamma u_{xxx} = 0, \tag{18}$$

$$v_t + \alpha u^n v_x + \beta v_{xx} + \gamma v_{xxx} = 0. \tag{19}$$

The third-order Lagrangian for this system of equations (18)-(19) is

$$L = v(u_t + \alpha u^n u_x - \beta u_{xx} + \gamma u_{xxx}). \tag{20}$$

Now using the conservation theorem Ibragimov (2007), the conserved vector components are obtained using

$$T^i = \xi L + W \frac{\delta L}{\delta u_i} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W) \frac{\delta L}{\delta u_{i_1 \dots i_s}}, \tag{21}$$

where L is the Lagrangian given in (20) and $W = \eta - \xi^1 u_t - \xi^2 u_x$ is the Lie characteristic function.

The KdVB equation (1) has two translation symmetries given in (3). The Lie characteristic function corresponding to the time-translation symmetry $X_1 = \partial/\partial t$ is $W = -u_t$. Applying the conservation theorem Ibragimov (2007), the components of the conservation law of energy associated with X_1 are

$$T_1^t = \alpha v u^n u_x - \beta v u_{xx} + \gamma v u_{xxx},$$

$$T_1^x = -\alpha v u^n u_t - \beta v_x u_t - \gamma v_{xx} u_t + \beta v u_{xt} + \gamma v_x u_{xt} - \gamma v u_{xxt}.$$

In the same manner, the Lie characteristic function corresponding to the space-translation symmetry $X_2 = \partial/\partial x$ is $W = -u_x$. Hence, the associated conservation law of linear momentum, has conserved vector

$$\begin{aligned} T_2^t &= -v u_x, \\ T_2^x &= v u_t - \beta v_x u_x - \gamma v_{xx} u_x + \gamma v_x u_{xx}. \end{aligned}$$

4. Concluding Remarks

In this paper we studied the Korteweg-de Vries Burgers equation with power law nonlinearity (1). We presented exact solutions of (1) using Lie symmetry analysis along with Kudryashov's method. The solutions obtained were travelling wave solutions. Furthermore, we constructed conservation laws using the new conservation theorem due to Ibragimov.

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